A comparison of the inverse scattering series direct non-linear inversion and the iterative linear inversion for parameter estimation across a single horizontal reflector

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85rd International Annual Meeting, SEG, Expanded Abstracts (2015

Abstract

The inverse scattering series (ISS) can achieve all seismic processing objectives directly without requiring any subsurface information. There are isolated task-specific subseries that derived from the ISS, which can perform free-surface multiple removal, internal multiple removal, depth imaging, parameter estimation, and Q compensation. Each isolated subseries assumes that only one task is performed. In this report, we will focus on the parameter estimation subseries and reduce it to a 1D normal incidence wave on a 1D acoustic earth where a single measured pressure wave is the input data. Under that very limited and focused circumstance, we are examining the difference between the iterative linear inverse and the direct inverse represented by the ISS parameter estimation subseries. A direct comparison is realizable in this specific example. The iterative approach shown in this example doesn't incorporate practical issues, e.g., the numerical noise and the different generalized inverses at each step. However, the ISS method performs as it does in practice with an analytic and unchanged inverse at every step. The comparison tests their convergence and the rate of convergence for different velocity contrasts. The rate of convergence of the ISS inversion method is analytically and numerically studied. When the reflection coefficient $R < 0.618$, the ISS inversion subseries monotonically term-by-term improves the estimation of medium properties; when $R > 0.618$, the ISS inversion subseries still converges, but not monotonically. Numerical tests show that when the velocity contrast is small, both inversion methods converge and the ISS inversion method converges faster than the iterative inversion method. When the velocity contrast increases, the iterative inversion method

can not be computable and the ISS inversion method always converges. Therefore, for the simplest situation, the iterative linear inversion is not equivalent to the ISS direct non-linear solution. For more complicated circumstances, the difference is much greater, not just on the algorithms, but also on data requirements.

1 Introduction

The objective of seismic inversion is to estimate the medium properties of the subsurface from the recorded wavefield at the surface. Inversion methods can be classified as a direct method or an indirect method. A direct inversion method can solve an inverse problem (as its name suggests) directly depending on the algorithm and its data requirements without searching or model matching. On the other hand, an indirect inversion method solves the inverse problem through indirect ways (Weglein, 2015a): (1) model matching, (2) objective/cost functions, (3) searching algorithms, (4) iterative linear inversion, and (5) methods corresponding to necessary but not sufficient conditions, e.g., common image gather flatness. For example, a quadratic equation $ax^2+bx+c=0$ can be solved through a direct method as $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, or it can be solved by an indirect method searching for x such that $(ax^2+bx+c)^2$ is a minimum.

A direct inverse solution for parameter estimation can be derived from an operator identity that relates the change in a medium's properties and the commensurate change in the wavefield. This operator identity is a general inversion methodology and can accommodate any model-type, for example, acoustic, elastic, anisotropic, heterogeneous, inelastic. The direct inverse solution is in the form of a series, referred to as the inverse scattering series (Weglein et al., 2003). It can achieve all processing objectives within a single framework without requiring any subsurface information. There are isolated-task inverse scattering subseries derived from the ISS, which can perform free-surface multiple removal, internal multiple removal, depth imaging, parameter estimation, and Q compensation. The direct inverse solution (Weglein et al., 2003; 2009) provides a solid framework and firm math-physics foundation for the data requirements and algorithms to solve the inverse problem. For an elastic heterogeneous medium, Zhang and Weglein (2006) shows that the direct inverse requires multi-component/PS (P-component and S-component) data and prescribes how that data are utilized for a direct parameter estimation solution.

In this paper, we focus on analyzing and examining the ISS inversion subseries for parameter estimation. The distinct issues of: (1) data requirements, (2) model-type, and (3) inversion algorithm for the direct inverse are all important (Weglein, 2015b). For a normal incident wave on a single horizontal reflector in an acoustic medium, we can isolate and focus on the algorithmic difference when mode-type agrees and there is the same data, a single reflector and acoustic P wave. Under that very limited and focused circumstance, a direct comparison is realizable and the iterative approach doesn't incorporate practical issues, e.g., the numerical noise and the different generalized inverses at each step. However, the ISS method performs as it would in practice with an analytic and unchanged inverse at every step. The numerical results show a comparison between the ISS direct non-linear inversion and the iterative inversion (Yang and Weglein, 2015) on a 1D one parameter model with a single horizontal reflector, where the velocity is assumed to be known above the reflector and unknown below the reflector. Their convergence and the rate of convergence will be discussed and studied. In the ISS inversion subseries, each term of the series works towards the final goal. Sometimes when more terms in the series are included, the estimation may be worse locally, but in fact it is purposeful and essential in the contribution towards convergence and the final goal. This property has also been indicated by Carvalho (1992) in the free-surface multiple elimination subseries, e.g., what appears to make a second-order free-surface multiple larger with a first-order free-surface algorithm is actually preparing the second-order multiple to be removed by the higher-order terms. This simple example provides a guide when we move on to the more complicated elastic world.

The report is arranged as follows: First, the ISS direct inversion method is discussed. Second, the direct inversion is presented in the 2D heterogeneous elastic medium. Third, the ISS direct inversion and the iterative linear inversion are examined and compared in a 1D acoustic medium. Finally, we offer a discussion and conclusions.

2 Theory

The direct inverse solution (Weglein et al., 2003; Zhang, 2006) is derived from the operator identity that relates the change in a medium's properties and the commensurate change in the wavefield. Let L_0, G_0, L, and G be the differential operators and Green's functions for the reference and actual media, respectively, that satisfy:

$$L_0 G_0 = \delta \quad \text{and} \quad LG = \delta,$$

where δ is a Dirac δ-function. We define the perturbation operator, V, and the scattered wavefield, ψ_s, as follows:

$$V \equiv L_0 - L \quad \text{and} \quad \psi_s \equiv G - G_0.$$

2.1 The operator identity

The relationship (called the Lippmann-Schwinger or scattering theory equation)

$$G = G_0 + G_0 V G \tag{1}$$

is an operator identity that follows from

$$L^{-1} = L_0^{-1} + L_0^{-1}(L_0 - L)L^{-1}.$$

For forward modeling the wavefield, G, for a medium described by L is given by

$$L \to G \quad \text{or} \quad L_0, V \to G$$

where the second form has L entering the modeling algorithms in terms of L_0 and V. Modeling using scattering theory requires a complete and detailed knowledge of medium properties.

2.2 Direct forward series and direct inverse series

The operator identity equation 1 can be solved for G as

$$G = (1 - G_0 V)^{-1} G_0, \tag{2}$$

or

$$G = G_0 + G_0 V G_0 + G_0 V G_0 V G_0 + \dots \quad . \tag{3}$$

Equation 3 has the form of a generalized geometric series

$$G - G_0 = S = ar + ar^2 + \cdots = \frac{ar}{1-r}, \tag{4}$$

where we identify $a = G_0$ and $r = VG_0$ in equation 3, and

$$S = S_1 + S_2 + S_3 + \dots \quad . \tag{5}$$

The portions of S that are linear, quadratic, ... in r are:

$$\begin{aligned} S_1 &= ar, \\ S_2 &= ar^2, \\ &\vdots \end{aligned}$$

and the sum is

$$S = \frac{ar}{1-r}. \tag{6}$$

Solving equation 6 for r, produces the inverse geometric series,

$$\begin{aligned} r &= \frac{S/a}{1+S/a} = S/a - (S/a)^2 + (S/a)^3 + \dots \\ &= r_1 + r_2 + r_3 + \dots . \end{aligned}$$

This is the simplest prototype of an inverse series, i.e., the inverse of the geometric series. For the seismic inverse problem, we associate S with the measured data

$$S = (G - G_0)_{ms} = \text{Data},$$

and the forward and inverse series follow from treating the forward solution as S in terms of V, and the inverse solution as V in terms of S. The inverse series assumes

$$V = V_1 + V_2 + V_3 + \dots, \tag{7}$$

where V_n is the portion of V that is n^{th} order in the data. The identity (equation 1) provides a geometric forward series rather than a Taylor series. In general, a Taylor series doesn't have an inverse series; however, a geometric series has an inverse series. All conventional current mainstream inversion methods, including iterative linear inversion and FWI, are based on a Taylor series concept. Solving a forward problem in an inverse sense is not the same as solving an inverse problem directly.

In terms of the expansion of V in equation 7, and G_0, G, $D = (G - G_0)_{ms}$, the inverse scattering series (Weglein et al., 2003) can be obtained as

$$
\begin{aligned}
G_0V_1G_0 =& D, && (8)\\
G_0V_2G_0 =& - G_0V_1G_0V_1G_0, && (9)\\
G_0V_3G_0 =& - G_0V_1G_0V_1G_0V_1G_0 \\
& - G_0V_1G_0V_2G_0 - G_0V_2G_0V_1G_0, && (10)\\
& \vdots
\end{aligned}
$$

The inverse scattering series provides a direct method for obtaining the subsurface information by inverting the series order-by-order to solve for the perturbation operator V, using only the measured data D and a reference Green's function G_0, for any type of medium.

2.3 The operator identity in a 2D heterogeneous elastic medium

The method for a 2D elastic heterogeneous earth is exemplified. The starting point for the 3D generalization is found in Stolt and Weglein (2012). The 2D elastic wave equation for a heterogeneous isotropic medium (Zhang, 2006) is

$$
L\vec{u} = \begin{pmatrix} f_x \\ f_z \end{pmatrix} \quad \text{and} \quad \hat{L} \begin{pmatrix} \phi^P \\ \phi^S \end{pmatrix} = \begin{pmatrix} F^P \\ F^S \end{pmatrix}, \tag{11}
$$

where $\vec{u}$, f_x, and f_z are the displacement and forces in displacement coordinates and ϕ_P, ϕ_S and F^P, F^S are the P and S waves and the force components in P and S coordinates. The operators L, L_0 and V are

$$L = \left[\rho\omega^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \partial_x\gamma\partial_x + \partial_z\mu\partial_z & \partial_x(\gamma-2\mu)\partial_z + \partial_z\mu\partial_x \\ \partial_z(\gamma-2\mu)\partial_x + \partial_x\mu\partial_z & \partial_z\gamma\partial_z + \partial_x\mu\partial_x \end{pmatrix}\right],$$

$$L_0 = \left[\rho\omega^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \gamma_0\partial_x^2 + \mu_0\partial_z^2 & (\gamma_0-\mu_0)\partial_x\partial_z \\ (\gamma_0-\mu_0)\partial_x\partial_z & \mu_0\partial_x^2 + \gamma_0\partial_z^2 \end{pmatrix}\right],$$

$$V \equiv L_0 - L = \begin{bmatrix} a_\rho\omega^2 + \alpha_0^2\partial_x a_\gamma\partial_x + \beta_0^2\partial_z a_\mu\partial_z & \partial_x(\alpha_0^2 a_\gamma - 2\beta_0^2 a_\mu)\partial_z + \beta_0^2\partial_z a_\mu\partial_x \\ \partial_z(\alpha_0^2 a_\gamma - 2\beta_0^2 a_\mu)\partial_x + \beta_0^2\partial_x a_\mu\partial_z & a_\rho\omega^2 + \alpha_0^2\partial_z a_\gamma\partial_z + \beta_0^2\partial_x a_\mu\partial_x \end{bmatrix},$$

where the quantities $a_\rho \equiv \rho/\rho_0 - 1$, $a_\gamma \equiv \gamma/\gamma_0 - 1$, $a_\mu \equiv \mu/\mu_0 - 1$ are defined in terms of the bulk modulus, shear modulus and density (γ_0, μ_0, ρ_0, γ, μ, ρ) in the reference and actual media, respectively.

The forward problem is found from the identity equation 3 and the elastic wave equation 11 (in PS coordinates) as

$$\begin{aligned} &\hat{G} - \hat{G}_0 = \hat{G}_0\hat{V}\hat{G}_0 + \hat{G}_0\hat{V}\hat{G}_0\hat{V}\hat{G}_0 + \ldots \\ &\begin{pmatrix} \hat{D}^{PP} & \hat{D}^{PS} \\ \hat{D}^{SP} & \hat{D}^{SS} \end{pmatrix} = \begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix}\begin{pmatrix} \hat{V}^{PP} & \hat{V}^{PS} \\ \hat{V}^{SP} & \hat{V}^{SS} \end{pmatrix}\begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix} \\ &+ \begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix}\begin{pmatrix} \hat{V}^{PP} & \hat{V}^{PS} \\ \hat{V}^{SP} & \hat{V}^{SS} \end{pmatrix}\begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix}\begin{pmatrix} \hat{V}^{PP} & \hat{V}^{PS} \\ \hat{V}^{SP} & \hat{V}^{SS} \end{pmatrix}\begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix} + \ldots \end{aligned} \tag{12}$$

and the inverse solution, equations 8-10, for the elastic equation 11 is

$$\begin{aligned} &\begin{pmatrix} \hat{D}^{PP} & \hat{D}^{PS} \\ \hat{D}^{SP} & \hat{D}^{SS} \end{pmatrix} = \begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix}\begin{pmatrix} \hat{V}_1^{PP} & \hat{V}_1^{PS} \\ \hat{V}_1^{SP} & \hat{V}_1^{SS} \end{pmatrix}\begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix}, \\ &\begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix}\begin{pmatrix} \hat{V}_2^{PP} & \hat{V}_2^{PS} \\ \hat{V}_2^{SP} & \hat{V}_2^{SS} \end{pmatrix}\begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix} \\ &= -\begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix}\begin{pmatrix} \hat{V}_1^{PP} & \hat{V}_1^{PS} \\ \hat{V}_1^{SP} & \hat{V}_1^{SS} \end{pmatrix}\begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix}\begin{pmatrix} \hat{V}_1^{PP} & \hat{V}_1^{PS} \\ \hat{V}_1^{SP} & \hat{V}_1^{SS} \end{pmatrix}\begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix}, \end{aligned} \tag{13}$$

where $\hat{V}^{PP} = \hat{V}_1^{PP} + \hat{V}_2^{PP} + \hat{V}_3^{PP} + \ldots$ and any one of the four matrix elements of V requires the four components of the data

$$\begin{pmatrix} \hat{D}^{PP} & \hat{D}^{PS} \\ \hat{D}^{SP} & \hat{D}^{SS} \end{pmatrix}.$$

In summary, from equation 12, $\hat{D}^{PP}$ can be determined in terms of the four elements of V and $\hat{V}^{PP}$, $\hat{V}^{PS}$, $\hat{V}^{SP}$, or $\hat{V}^{SS}$ require the four components of D. That's what the general relationship $G = G_0 + G_0 V G$ requires, i.e., a direct non-linear inverse solution is a solution order-by-order in the four matrix elements of D (in 2D).

2.4 Direct inverse and indirect inverse

The direct solution is not iterative linear inversion. Iterative linear inversion starts with equation 8. We solve for V_1 and change the reference medium iteratively. The new differential operator L'_0 and the new reference medium G'_0 satisfy

$$L'_0 = L_0 - V_1 \quad \text{and} \quad L'_0 G'_0 = \delta. \tag{14}$$

Through the same equation 8 with different reference background

$$G'_0 V'_1 G'_0 = D' = (G - G'_0)_{ms}, \tag{15}$$

where V'_1 is the portion of V linear in the data $(G - G'_0)_{ms}$. We can continually update L'_0 and G'_0, and finally solve for the perturbation operator V. The direct inverse solution equations 7 and 13 calls for a single unchanged reference medium, for computing $V_1, V_2, \dots$. For a homogeneous reference medium they are obtained by an analytic inverse. The inverse to find V_1 from data, is the same inverse to find $V_2, V_3, \dots,$ from equations 8-10. There are no numerical inverses, no generalized inverses, no inverses of matrices that contain noisy bandlimited data.

The difference between iterative linear and the direct inverse of equation 13 is much more substantive and serious than merely a different way to solve $G_0 V_1 G_0 = D$, equation 8, for V_1. If equation 8 is our entire basic theory, you can mistakenly think that $\hat{D}^{PP} = \hat{G}_0^P \hat{V}_1^{PP} \hat{G}_0^P$ is sufficient to update $\hat{D}^{PP} = \hat{G}_0'^P \hat{V}_1'^{PP} \hat{G}_0'^P$. This step loses contact with and violates the basic operator identity $G = G_0 + G_0 V G$ for the elastic wave equation. That's as serious as considering problems involving a right triangle and violating the Pythagorean theorem in your method. That is, iteratively updating PP data with an elastic model violates the basic relationship between changes in a medium, V and changes in the wavefield, $G - G_0$ for the simplest elastic earth model.

This direct inverse method provides a platform for amplitude analysis, AVO and FWI. It communicates when a "FWI" method should work, in principle. Iteratively inverting multi-component data has the correct data but doesn't corresponds to a direct inverse algorithm. To honor $G = G_0 + G_0VG$, you need both the data and the algorithm that direct inverse prescribes. Not recognizing the message that an operator identity and the elastic wave equation unequivocally communicate is a fundamental and significant contribution to the gap in effectiveness in current AVO and FWI method and application (equation 13). This analysis generalizes to 3D with P, S_h, and S_v data.

There's a role for direct and indirect methods in practical real world applications. Indirect methods are to be called upon for recognizing that the world is more complicated than the physics that we assume in our models and methods. For the part of the world that you are capturing in your model (and methods) nothing compares to direct methods for clarity and effectiveness. An optimal indirect method would seek to satisfy a cost function that derives from a property of the direct method. In that way the indirect and direct method would be aligned and cooperative for accommodating the part of the world described by your physical model and the part that is outside.

2.5 The operator identity in a 1D acoustic medium

Considering a simple 1D case, the model consists of two half-spaces with acoustic velocities c_0 and c_1 and an interface located at $z = a$ as shown in Figure 1. If we put the source and receiver on the surface, the pressure wave $P(t) = R\delta(t - 2a/c_0)$ will be recorded, where the reflection coefficient $R = \frac{c_1 - c_0}{c_1 + c_0}$. Without considering the imaging issue, R is the only input to the ISS and the iterative inversion methods. Choosing an acoustic whole-space with velocity c_0 as the reference medium, the perturbation V (Weglein et al., 2003) can be expanded as

$$V(z) = \frac{\omega^2}{c_0^2} - \frac{\omega^2}{c^2(z)} = \frac{\omega^2}{c_0^2}(1 - \frac{c_0^2}{c^2(z)}) = k_0^2\alpha(z), \tag{16}$$

where ω is the angular frequency, $c(z)$ is the local acoustic velocity, $k_0 = \omega/c_0$, and $\alpha(z) \equiv 1 - \frac{c_0^2}{c^2(z)}$. Depending on V, $\alpha(z)$ can be expanded as a series in terms of data, $\alpha(z) = \alpha_1(z) + \alpha_2(z) + \alpha_3(z) + \cdots$. Thus, we have

$$V_1 = k_0^2\alpha_1, \quad V_2 = k_0^2\alpha_2, \quad \cdots . \tag{17}$$

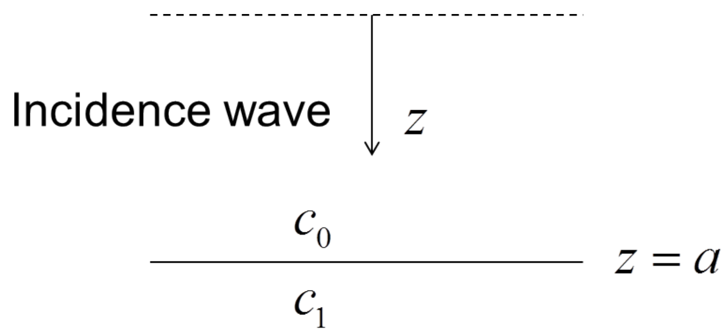

Figure 1: 1D acoustic model with velocities c_0 over c_1

From the inverse scattering series (Equations 8-10), Shaw et al. (2004) isolated the leading order imaging subseries and the direct non-linear inversion subseries.

In this section, we will focus on studying the convergence properties of the ISS inversion subseries. The inversion only terms isolated from the inverse scattering series (Zhang, 2006; Li, 2011) are

$$\alpha(z) = \alpha_1(z) - \frac{1}{2}\alpha_1^2(z) + \frac{3}{16}\alpha_1^3(z) + \cdots . \tag{18}$$

For a 1D normal incidence case, we invert G_0 of the linear equation (8) and obtain,

$$\alpha_1(z) = 4\int_{-\infty}^{z} D(z')dz', \tag{19}$$

where $z' = c_0 t/2$. Inserting data D gives

$$\alpha_1 = 4R, \tag{20}$$

when $z > a$, where the reflection coefficient $R = \frac{c_1 - c_0}{c_1 + c_0}$. Substituting α_1 into equation (18), the ISS direct non-linear inversion subseries in terms of R can be written as

$$\alpha = 4R - 8R^2 + 12R^3 + \cdots = 4R\sum_{n=0}^{\infty}(n+1)(-R)^n. \tag{21}$$

After solving for α, the inverted velocity $c(z)$ can be obtained through $c_1 = c_0/\sqrt{1-\alpha}$ (equation 16).

Considering the convergence property of the series for α or the inversion subseries, we can calculate the ratio test,

$$\left|\frac{\alpha_{n+1}}{\alpha_n}\right| = \left|\frac{(n+2)(-R)^{n+1}}{(n+1)(-R)^n}\right| = \left|\frac{n+2}{n+1}R\right|. \tag{22}$$

If $\lim_{n\to\infty}\left|\frac{\alpha_{n+1}}{\alpha_n}\right| < 1$, this subseries converges absolutely. That is

$$|R| < \lim_{n\to\infty}\frac{n+1}{n+2} = 1. \tag{23}$$

Therefore, the ISS direct non-linear inversion subseries converges when the reflection coefficient $|R|$ is less than 1, which is always true. Hence, for this example, the ISS inversion subseries will converge under any velocity contrasts between the two media.

For the iterative linear inversion, we will update the reference velocity $c_0' = c_0/\sqrt{1-\alpha_1}$ by using $\alpha_1 = 4R$. Then, the new linear inversion velocity is calculated by $\alpha_1' = 4R'$, where $R' = \frac{c_1 - c_0'}{c_1 + c_0'}$. The same procedure will be applied iteratively until we achieve the final inversion result.

3 Numerical examples for the 1D acoustic case

Numerical examples for the 1D acoustic medium case are shown in this section. First, an analytic example for the rate of convergence of the ISS inversion subseries is examined and studied for a 1D normal incidence case. Second, the convergence of the ISS direct inversion and iterative inversion are examined and compared.

3.1 Analytic example

The rate of convergence of the estimated α or the ISS inversion subseries (equation 21) is analytically examined and studied for a 1D normal incidence case. Since α is always convergent when $R < 1$, the summation of this subseries (Zhang, 2006) is

$$\alpha = 4R\sum_{n=0}^{\infty}(n+1)(-R)^n = 4R\frac{1}{(1+R)^2}. \tag{24}$$

If the error between the estimated and the actual α is monotonically decreasing, it means the subseries is a term-by-term added value improvement towards determining the actual medium properties. If this error is increasing before decreasing, it means that the estimation becomes worse before it gets better. In fact, locally worse is purposeful and essential in the contribution towards convergence and the final goal. In other words, the error for the first order and the error for the second order have the relation,

$$|\alpha - \alpha_1 - \alpha_2| > |\alpha - \alpha_1|, \tag{25}$$

i.e.,

$$|4R\frac{3R^2 + 2R^3}{(1+R)^2}| > |4R\frac{-R^2 - 2R}{(1+R)^2}|. \tag{26}$$

After simplification, it gives

$$R^2 + R - 1 > 0. \tag{27}$$

We can solve it and obtain the reflection coefficient $R < \frac{-1-\sqrt{5}}{2} = -1.618$ or $R > \frac{-1+\sqrt{5}}{2} = 0.618$. Therefore, when $R > 0.618$, the error increases first. Similarly, if the error for the third order is greater than that for the second order, we get $R > 0.667$. If the error for the fourth order is greater than that for the third order, we obtain $R > 0.721$. In summary, when $R > 0.618$ the error increases and the estimated α gets worse before getting better. The dashed green line in Figure 2 shows that when the reflection coefficient R is equal to 0.618, the error for the first order is equal to the error for the second order. The detail of the numerical tests will be discussed in the next section.

3.2 Numerical tests

In this section, we will examine the convergence property and the rate of convergence of α by using the ISS inversion subseries (equation 21) and the iterative linear inversion methods for the velocity contrast in the 1D acoustic case. In addition, the inversion results by these two methods are discussed and compared.

In the simple 1D model (Figure 1), only one parameter (velocity) varies and a plane wave propagates into the medium. There is only a single reflector and we assume the velocity is known above the reflector and unknown below the reflector. We will examine and compare

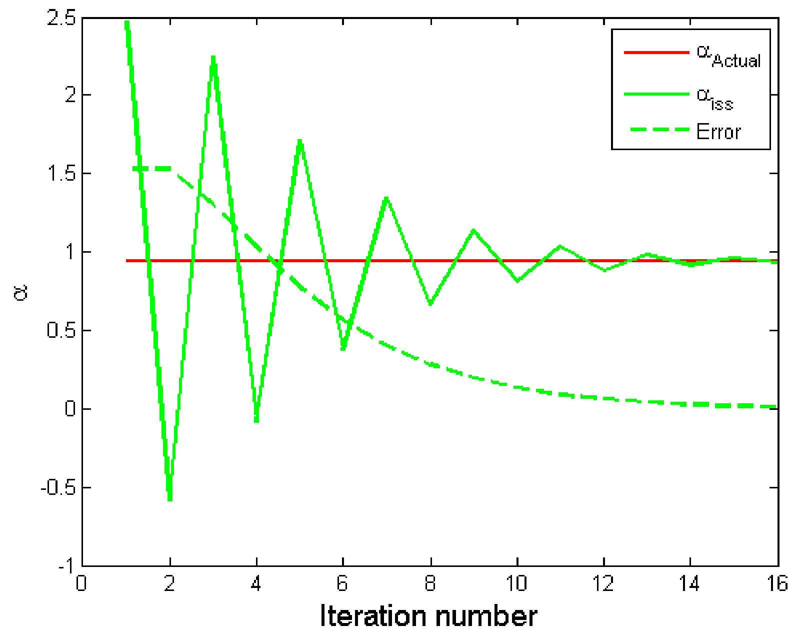

Figure 2: The error (dashed green line) of estimated α at $R = 0.6180$ and $\alpha = 0.9443$.

the convergence of the perturbation α and the inversion results by using the ISS direct non-linear method and the iterative linear method. In this model, we set the reference velocity $c_0 = 1500m/s$ and lower half space velocity $c_1 = 2000m/s$.

Figure 3 shows the estimated α by the ISS method (green line) and the iterative inversion method (blue line). The red line represents the actual α that is calculated from our model. The horizontal axis represents the order of the ISS inversion subseries or the iteration numbers. The vertical axis shows the value of α. From the estimated α, it can be seen that at the small velocity contrast, the estimated α by ISS method becomes the actual α after about five orders calculation and the estimated α by the iterative inversion method goes to zero as we expected, because after several iteration, the updated model is close to and approaching to the actual model. Figure 4 represents the velocity estimation. We can see that both methods converge and produce correct velocity after five orders or iterations. From both Figures 3 and 4, we can see that both methods converge very fast at the small velocity contrast and the ISS method converge faster than the iterative inversion method.

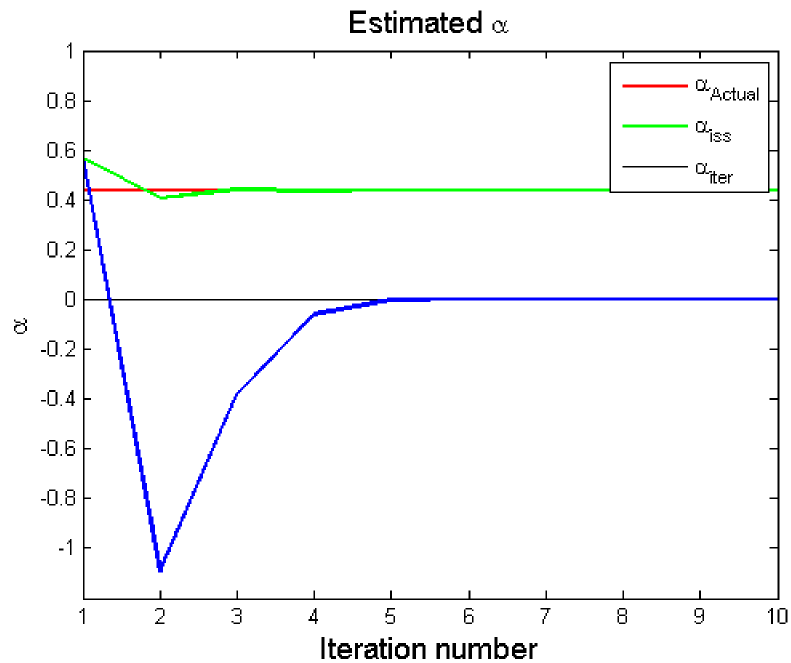

Figure 3: The estimated α at $R = 0.1429$: The horizontal axis is the order of the ISS suberies or the iteration numbers, and the vertical axis shows the value of α. The red line shows the actual value of $\alpha = 0.4375$. The green and blue lines show the estimations of α by using the ISS inversion method and the iterative inversion method.

When the velocity contrast is getting bigger, the iterative inversion method can not be computable in this example, but the ISS inversion method always converges and produces correct results (see green line in Figures 5) after the summation of more orders of α.

From the dashed green line in Figures 3 and 5, at the small contrasts, the error between the estimated and the actual α is monotonically decreasing, in other words, the estimation of α is always a term-by-term added value improvement towards determine c_1; when the contrast increasing (Figure 2), the error is not monotonic. The estimation of α can be worse before it gets better. However, when it starts to add value, it is getting better when each further term is added to the series.

As the analytic calculation, when the reflection coefficient R is smaller than 0.618, this inversion subseries gives a monotonically term-by-term added value improvement towards

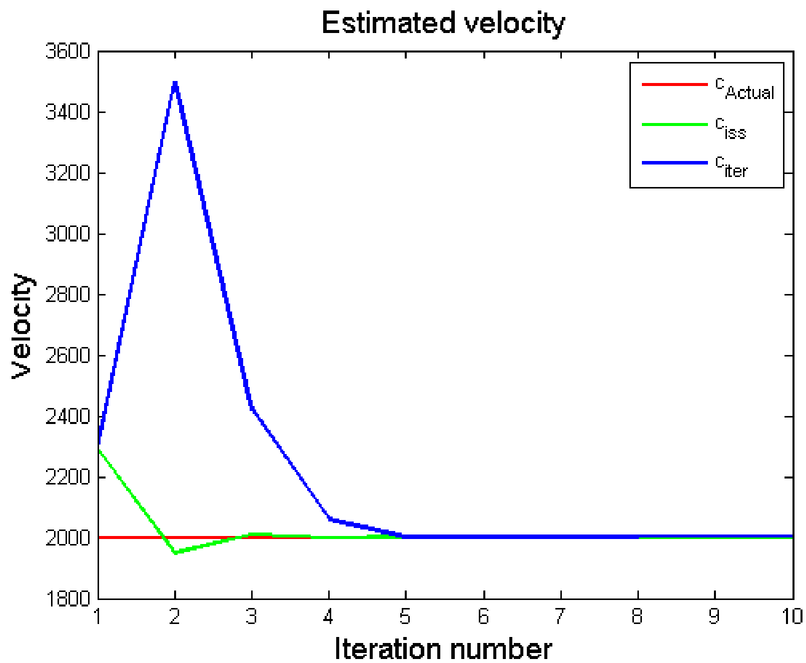

Figure 4: The estimated velocity by using the ISS inversion method (green line) and the iterative inversion method (blue line).

determining c_1. When the reflection coefficient R is equal to 0.618, the error becomes flat as shown in Figure 2. When the reflection coefficient is larger than 0.618, the series still converges, but the estimation of α will become worse before it gets better. From the analytic and numerical examples, we can see that each term in the series works towards the final goal. Sometimes when more terms in the series are included, the estimation looks worse locally, but once it starts to improve the estimation at a specific order, the approximations never become worse again, every single term after that order will produce an improved estimation.

As we know, the reflection coefficient R is almost always less than 0.2 in practice, so that both the ISS method and the iterative method converge, but the ISS method converges faster than the iterative method. Moreover, for more complicated circumstances (e.g., the elastic non-normal incidence case), the difference between the ISS method and the iterative is much greater, not just on the algorithms, but also on data requirements and on how the band-limited noisy nature of the seismic data impact the inverse operators in the iterative

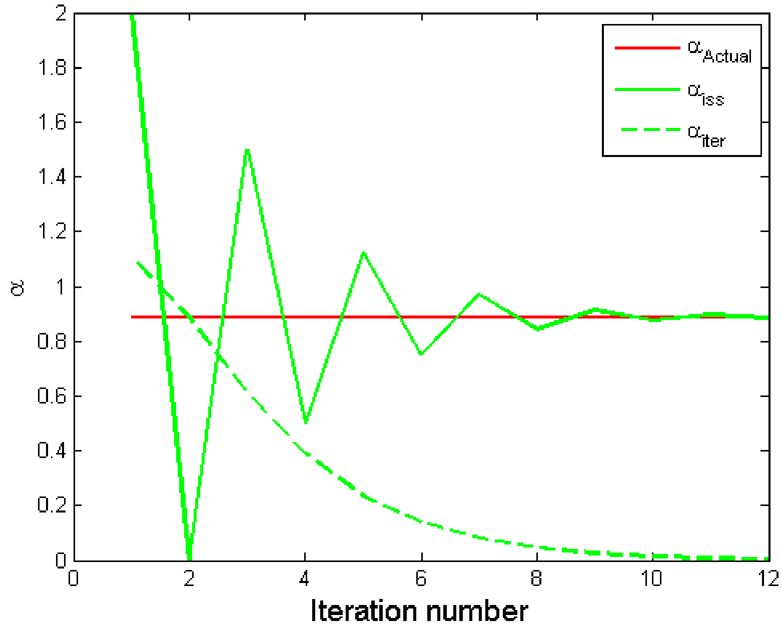

Figure 5: The estimated α at $c_1 = 4500m/s$ and $R = 0.5000$ by using the ISS inversion method.

method but not in the ISS method.

4 Conclusions

In this report, we discuss a direct inverse method, which is derived from the operator identity that relates the change in a medium's properties and the commensurate change in the wave-field. We describe the direct inversion algorithm for parameter estimation (ISS subseries) and its data requirements. In a specific 1D acoustic medium, we examine and compare the ISS direct non-linear inversion and the iterative inversion for parameter estimation across a single horizontal reflector, where the velocity is assumed to be known above the reflector and unknown below the reflector. The rate of convergence of the ISS inversion method is analytically and numerically studied. From the analytic example, we show that when the reflection coefficient $R < 0.618$, the ISS inversion subseries is a term-by-term improvement towards determining medium properties; when $R > 0.618$, the inversion subseries still converges, but the estimation will locally be less accurate before it converges. Numerical results show that when the velocity contrast is small, i.e., the reflection coefficient is small, both inversion methods converge and the ISS inversion method converges faster than the iterative inversion method. When velocity contrast increases, the reflection coefficient gets larger, the iterative inversion method can not be computable and the ISS inversion method always converges. Hence, for the simplest situation, the iterative linear inversion is not equivalent to the direct non-linear solution provided by the inverse scattering series. For more complicated circumstances (e.g., the elastic non-normal incidence case), the difference is much greater, not just on the algorithms, but also on data requirements and on how the band-limited noisy nature of the seismic data impact the inverse operators in iterative linear inversion but not in the ISS direct inversion.

5 ACKNOWLEDGMENTS

The author is grateful to all M-OSRP sponsors for their support and encourage of this research.

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